Stability criterion for stationary bound states of solitons with radiationless oscillating tails

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We investigate stationary bound states of bright solitary waves in an optical fiber with fourth-order dispersion, which is described by a generalized nonlinear Schrödinger equation with an additional fourth-order derivative term. It is shown that various families of two-soliton (and multisoliton) bound states exist for this equation in the parameter region where single solitons have radiationless oscillatory tails. We analyze and discuss their stability and possible applications. A stability criterion for stationary two-soliton (and multisoliton) states of a conservative Hamiltonian dynamical system is derived.

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I. INTRODUCTION

A crucial issue in time-division multiplexed optical transmissions is to maintain a constant temporal separation between the individual pulses that carry information. In soliton-based high bit rate transmission systems, the relative position of the individual pulses in a sequence can be scrambled by the Gordon-Haus effect [1] and/or by soliton interactions [2]. Forcing each soliton to remain in its own time slot may be achieved by distributing bandpass filters [3–5] or synchronous amplitude modulators [6] along the line. In all of these methods, even though soliton interaction is suppressed, it is not reduced to zero.

Here we are interested in developing a different strategy to prevent pulse coalescence. One of the ways to solve this problem is to use solitons with oscillating tails [7]. In this case the interaction between the solitons themselves can establish a certain minimal distance between the pulses and hence solitons can be separated from each other by some potential barrier. On the other hand, bound states of these solitons can exist. The question of the interaction of two (or more) solitary waves and the condition for the existence of soliton bound states in various dynamical systems has been formulated in [8–10] and is still a topic of active discussion (e.g., [11,12]). Bound states (BS's) of solitons exist when single-soliton solutions have nonmonotonic asymptotics (radiationless oscillating tails) [10]. These tails produce local extrema in an effective interaction potential of weakly overlapping solitons and therefore these solitons can trap each other at certain distances. Thus the single solitons can be bound into multisoliton states which, in principle, can serve as information flow in optical transmission lines. To transmit information, the train of solitons should be modulated by changing the relative phases between neighboring solitons in the train. However, the stability of the whole train, and specifically the stability of two-soliton BS's, is a question to be addressed before considering the practical use of this idea. In this paper, we investigate, analytically and numerically, stationary two-soliton (and multisoliton) BS's of bright solitons of the generalized nonlinear Schrödinger equation (NLSE) with an additional fourth-order derivative term and analyze their stability.

The remainder of the paper is organized as follows. In Sec. II we formulate the problem and (for completeness of this paper) remind the reader of the main results of [7,13]. In Sec. III we analytically determine stationary soliton BS's of the generalized NLSE. In Sec. IV we consider their stability and derive a stability criterion for stationary two-soliton (and multisoliton) BS's of any conservative Hamiltonian dynamical system. In Sec. V we present the results of our numerical analysis and compare them with the results of Secs. III and IV. Section VI contains our conclusions and a discussion.

II. STATEMENT OF THE PROBLEM

We start with the generalized NLSE with an additional fourth-order derivative term. In optics, this equation describes pulse propagation in fibers with fourth-order dispersion [7,13,14]. It can be obtained under the usual assumption of a slowly varying field envelope. In the dimensionless form it is given by

$$i\frac{\partial w}{\partial \xi}+\frac{\partial^2 w}{\partial \tau^2}-\varepsilon\frac{\partial^4 w}{\partial \tau^4}+|w|^2 w=0. \qquad (1)$$

Here $w(\xi,\tau)$ is the slowly varying pulse envelope, ξ is the normalized longitudinal coordinate, τ is the normalized retarded time, and ε is proportional to the ratio of fourth-order and second-order group velocity dispersions. In this paper we investigate only the case $\varepsilon>0$, since in the case $\varepsilon<0$ stationary single-soliton solutions do not exist (any single pulselike solution emits radiation [15]). Equation (1) describes the pulse propagation in a specially designed optical fiber. The group velocity versus wavelength dependence for such fibers has a maximum at the pulse central frequency. The way to design this type of fiber is discussed in [16] in detail. It is important to note that the value of the second-order dispersion can be very close to zero at the central frequency and thus it can be of the same order as the fourth-order dispersion [i.e., $\varepsilon \sim 1$ in Eq. (1)]. In this case the pulse duration can be chosen around $T = 1$ ps to ensure that even moderate pulse powers ($P \approx 1$ W) make the effects which we are discussing in this paper observable. Other terms which should be added to Eq. (1) when one investigates the propagation of the pulses with $T \leq 0.1$ ps (see [17]) are caused by different physical reasons and can be omitted in this work. Equation (1) is also related to other fields of physics (e.g., it describes propagation of "whistlers" in a plasma [18]).

We note that it is possible to reduce the number of parameters of Eq. (1) by using rescaling transformation: $t = \tau/\sqrt{\varepsilon}$, $x = \xi/\varepsilon$, and $u = w\sqrt{\varepsilon}$. These transformations turn Eq. (1) into

$$i\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial t^4} + |u|^2 u = 0. \tag{2}$$

Below we analyze Eq. (2), noting that one can easily get all solutions of Eq. (1) from the solutions of Eq. (2) using the inverse scaling transformations.

Equation (2) has three integrals of motion: the energy of the pulse

$$Q = \int_{-\infty}^{\infty} |u|^2 dt, \tag{3}$$

the momentum

$$M = i\int_{-\infty}^{\infty} (u_t u^* - u_t^* u) dt, \tag{4}$$

and the Hamiltonian

$$H = \int_{-\infty}^{\infty} (|u_t|^2 + |u_{tt}|^2 - \tfrac{1}{2}|u|^4) dt. \tag{5}$$

In this paper we are interested in solutions of Eq. (2) with $M = 0$. Hence only two of the conserved quantities (energy and Hamiltonian) are of importance. Equation (2) can be written in the canonical Hamiltonian form [19]

$$iu_x = \frac{\delta H}{\delta u^*}, \quad iu_x^* = -\frac{\delta H}{\delta u}, \tag{6}$$

where δ denotes variational derivative.

Equations (5) and (6) define a Hamiltonian dynamical system on an infinite-dimensional phase space of complex functions (u, u^*), which decrease to zero at infinity and can be analyzed using the theory of Hamiltonian systems.

Stationary pulselike solutions of Eq. (2) have the following form:

$$u(x,t) = y(q, t - t_0) e^{i(qx + \varphi_0)}, \tag{7}$$

where t_0 and φ_0 are the position of the soliton center and the initial phase of the soliton, respectively, q ($q > 0$) is the parameter of this one-soliton solution (nonlinearly induced shift to the wave number), and $y(q, t - t_0)$ is a real function of its parameters.

The equation for finding stationary solutions in a variational formulation can be written in the form

$$\delta(H - qQ) = 0. \tag{8}$$

A variational formulation of the problem (8) also defines the stability of stationary states [20]. For any fixed Q the stationary state is stable if the corresponding H has a local minimum, with q being the Lagrangian multiplier. The ordinary differential equation for finding stationary solutions obtained from Eq. (8) is

$$\frac{d^2 y}{dt^2} - \frac{d^4 y}{dt^4} + y^3 - qy = 0. \tag{9}$$

For a particular case $q = 0.16$ an exact solution of Eq. (9) has been found recently [13]:

$$y(t) = \frac{\pm\sqrt{\frac{3}{10}}}{\cosh^2[(t - t_0)/\sqrt{20}]}. \tag{10}$$

For other values of q the exact analytical form of solutions is unknown. Below in our analysis of soliton interactions we use an approximation which has been found using a variational approach [13]:

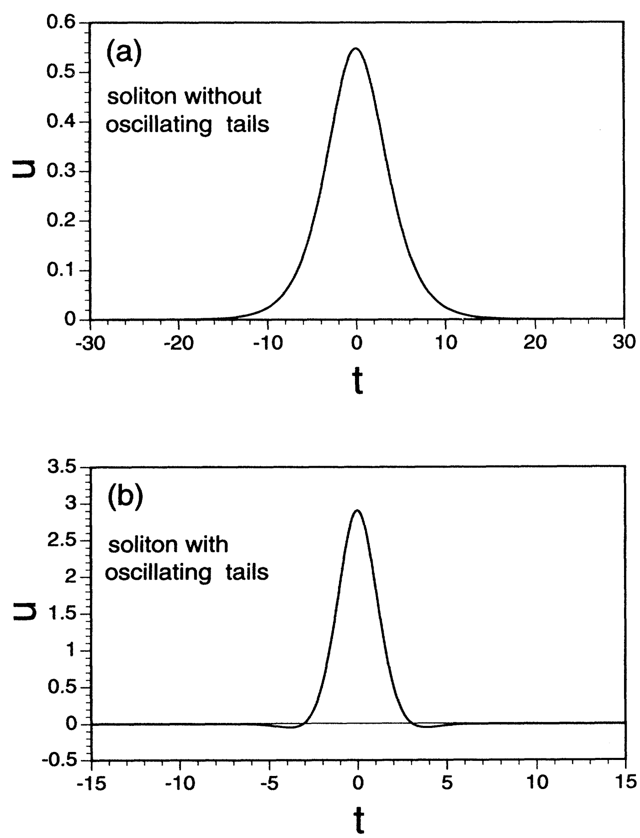

FIG. 1. Examples of single-soliton stationary solutions of Eq. (2). (a) Soliton without oscillating tails ($q = 0.16$, point A in Figs. 2 and 3). (b) Soliton with oscillating tails ($q = 5.0$, point B in Figs. 2 and 3).

$$u(x,t) = \frac{a(q)}{\cosh^2[k(q)(t-t_0)]} e^{i(qx+\varphi_0)}, \tag{11}$$

where $a(q)$ and $k(q)$ are given by

$$k(q) = \sqrt{\frac{5q}{6+\sqrt{36+400q}}},$$
$$a(q) = \pm\sqrt{\frac{14}{3}k^2(q) + \frac{80}{3}k^4(q)}. \tag{12}$$

Expressions (11) and (12) are a reasonable approximation of an actual one-soliton solution of Eq. (2) *in its central part* [i.e., for $k(q)(t-t_0) \sim 1$]. However, those expressions (11) and (12) do not have correct asymptotic behavior. Asymptotics becomes qualitatively wrong when $q > 0.25$. For this region of q the asymptotics can be obtained from analysis of the linearized Eq. (2):

$$\lim_{t\to\pm\infty} y(t,q) = 4a(q)e^{-\lambda(q)|t-t_0|}\cos[\omega(q)|t-t_0| + \psi_0(q)], \tag{13}$$

where $\lambda(q)$ and $\omega(q)$ are given by

$$\lambda(q) = q^{1/4}\cos\left(\frac{\arctan\sqrt{4q-1}}{2}\right),$$
$$\omega(q) = q^{1/4}\sin\left(\frac{\arctan\sqrt{4q-1}}{2}\right), \tag{14}$$

and $\psi_0(q) \approx \pi/2$. We recall that we use the approximate solution (11) and (12) *only* to describe single solitons *in their central part*, always taking into account the correct asymptotic form of *tails* (13) and (14).

Characteristic examples of single-soliton solutions of Eq. (2) without and with oscillating tails are given in Figs. 1(a) and 1(b) correspondingly. These solutions form one-parameter family with q as a parameter. Because radiationless oscillating tails of single-soliton solutions exist at $q > 0.25$, BS's of these solitons, i.e., stationary multisoliton solutions, exist only in the same region of q.

III. BOUND STATES OF SOLITONS

The Hamiltonian of any two interacting solitons which are located far enough from each other can be written in the approximate form

$$H = H_1 + H_2 + H_{\text{int}}, \tag{15}$$

where H_1 and H_2 are Hamiltonians of the individual solitons and H_{int} is a small interaction term, which is determined by the nonlinear part of Hamiltonian (5) (see [10])

$$H_{\text{NL}} = -\int_{-\infty}^{\infty} \frac{|u(t,x)|^4}{2} dt. \tag{16}$$

Now substituting $u = u_1 + u_2$, where u_1 and u_2 stand for unperturbed individual solitons, in H_{NL} and linearizing relative to u_2 (and u_1) we obtain

$$H_{\text{int}}(\Delta t, \Delta\varphi) = -2\int_{-\infty}^{\infty} |u_1(t,x)|^2 \text{Re}[u_1(t,x)u_2^*(t,x)]dt + (1 \leftrightarrow 2), \tag{17}$$

where the expression describing the interaction of the central part of the first soliton with the tail of the second one is written down explicitly. Because of the symmetry, the similar expression $(1 \leftrightarrow 2)$ has to be added for the interaction of the central part of the second soliton with the tail of the first one. The interaction part H_{int} (17) depends on the relative distance Δt between the centers of the two solitons and their relative phase difference $\Delta\varphi$.

Now, assuming that the two interacting solitons are identical and far separated and inserting Eqs. (11) and (12) (with $t_0 = 0$ and $\varphi_0 = 0$) for the central part of u_1 and Eqs. (13) and (14) (with $t_0 = \Delta t$ and $\varphi_0 = \Delta\varphi$) for the tail of u_2 in the first term on the right-hand side of the expression (17) [and making the similar substitutions for the remaining term $(1 \leftrightarrow 2)$] one can finally get

$$H_{\text{int}}(\Delta t, \Delta\varphi) = -A\cos(\Delta\varphi)e^{-\lambda(q)\Delta t} \times \cos[\omega(q)\Delta t + \psi_1(q)], \tag{18}$$

where $A \sim a^4/\lambda$ and $\psi_1(q) \approx \psi_0(q)$. Note that we have taken into account the correct asymptotic dependence of soliton tails (13) and (14) in calculating the expression (18).

In this paper we consider mainly interactions and BS's of *two* solitons. However, this approach can be extended for any number of solitons since, due to the exponential factor $e^{-\lambda(q)\Delta t}$ in H_{int}, only pair interactions between neighboring solitons are important.

BS's of two solitons exist if the interaction part of Hamiltonian (18) has local extrema. These extrema are determined by

$$\frac{\partial H_{\text{int}}}{\partial \Delta\varphi} = 0, \quad \frac{\partial H_{\text{int}}}{\partial \Delta t} = 0. \tag{19}$$

For every $q > 0.25$ there are two infinite sets of solutions of Eqs. (19) (i.e., there are two sets of families of BS's): the "symmetric" set (two solitons in phase) and the "antisymmetric" set (π-phase difference between solitons):

$$\Delta\varphi = 0, \pi,$$
$$\Delta t_n = \Delta t_1 + \frac{\pi}{\omega}(n-1), \tag{20}$$

where $n = 1, 2, 3, \ldots$ and $\Delta t_1 = [\pi - \psi_1(q) + \arccos(\lambda/\sqrt{\lambda^2+\omega^2})]/\omega$. For the symmetric set, Δt_n with odd (even) numbers correspond to local maxima (minima) of H_{int}. For the antisymmetric set of BS's the situation is opposite. Below, we call the stationary two-soliton BS for which the relative distance between partial solitons is Δt_n a "BS of nth order." We remind the reader that the BS's constructed above from single-soliton solutions have the same value of q as the individual solitons.

IV. STABILITY ANALYSIS

For the case of two identical, well-separated solitons, local extrema in H_{int} (18) result in extrema in the total Hamiltonian (15) at the same values of Δt_n and $\Delta\varphi$ for a fixed value of q. This, in turn, results in local extrema of the Hamiltonian (15) for a fixed Q. If the Hamiltonian has a local maximum, the corresponding BS is unstable with respect to a relative transverse shift of two solitons. If the Hamiltonian has a local minimum, then the corresponding BS is stable relative to this type of perturbation. However, the whole analysis was carried out for two identical solitons. Thus, even if the interaction part of Hamiltonian has a local minimum, other types of perturbations [including those which change the first two terms of the Hamiltonian (15)] must be considered before making a final conclusion about general stability.

Suppose that the Hamiltonian and energy for a family of single-soliton solutions are related by

$$H = f(Q), \tag{21}$$

where $f(Q)$ is some functional dependence, which can be found using approximate analytical methods or numerically. The Hamiltonian for the combined state of two identical solitons is given by

$$H = 2f(Q) + H_{\text{int}}(Q). \tag{22}$$

Now we consider perturbations that exchange energy between two solitons. Due to this perturbation, the energy Q_0 of one soliton is increased by the small amount ΔQ:

$$Q_1 = Q_0 + \Delta Q. \tag{23}$$

Since we are interested in perturbations that conserve total energy, the energy of the second soliton has to decrease by the same amount:

$$Q_2 = Q_0 - \Delta Q, \tag{24}$$

so that the change of the Hamiltonian is approximately given by

$$\Delta H = f''(Q_0)(\Delta Q)^2, \tag{25}$$

where $f''(Q) \equiv \frac{\partial^2 f}{\partial Q^2}$. The value of $H_{\text{int}}(Q)$ also depends on Q, but this dependence is relatively weak [since $H_{\text{int}}(Q)$ is exponentially small itself]. Without loss of generality, we can consider it as a constant in the vicinity of any particular $Q = Q_0$. Hence the sign of $f''(Q)$ at the point $(Q = Q_0)$ defines the stability of BS's. If $f''(Q)$ is negative, the Hamiltonian, which corresponds to the analyzed BS, does not have a local minimum for fixed $Q = Q_0$ so that this BS is unstable. For the family of one-soliton solutions of Eq. (2) $f''(Q)$ is negative for all possible values of Q (numerical result) and thus we can expect that two-soliton BS's are unstable.

The stability criterion derived above for our particular problem can be generalized: for any conservative nonintegrable systems having a family of single solitons with radiationless oscillating tails, the Hamiltonian versus energy curve for this family defines the stability of two-soliton BS's. The BS is stable if the second derivative of $f(Q)$ at the point of interest is positive and is unstable if the second derivative is negative.

All above derivations have been made in the approximation that the solitons are separated far enough from each other. When the solitons are close to each other (which is the most interesting case, because of large values of bound energy H_{int}), then formally we cannot apply our approach and need to use numerical methods. In the next section we compare our analytic results with

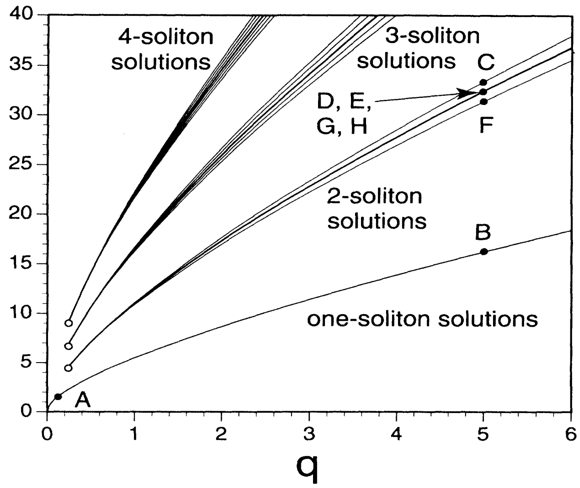

FIG. 2. Energy-dispersion diagram for the various families of soliton states. Each curve corresponds to a family of soliton solutions. Examples of soliton solutions corresponding to filled circles $(A-H)$ are shown in Figs. 1, 4, and 5. Open circles denote the starting points of the families of two-soliton (or multisoliton) BS's. Only curves corresponding to two-, three-, and four-soliton states are shown.

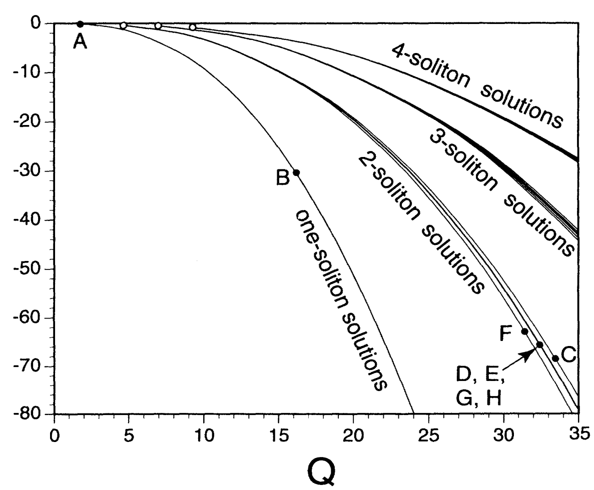

FIG. 3. Hamiltonian versus energy diagram for the various families of soliton states. The labels of points are the same as in Fig. 2. The sequence of points F, E, and C shows the correspondence between the variables H, Q, and q.

results of numerical simulations and discuss the BS stability question in detail.

V. NUMERICAL RESULTS

The results of Sec. III show that, in order to find stationary soliton solutions of Eq. (2), one should find localized solutions of Eq. (9). Equation (9) is a fourth-order nonlinear ordinary differential equation (ODE) with real parameters and it can be analyzed by means of the standard shooting technique (see [7] and [21]). The work [21] contains the general theorem showing the existence of multihump localized solutions in autonomous Hamiltonian systems of fourth-order (the ODE problem). The main results of our numerical analysis are presented in the form of an energy-dispersion diagram (see Fig. 2) and Hamiltonian versus energy diagram (see Fig. 3). In these diagrams Q and H are defined by the expressions

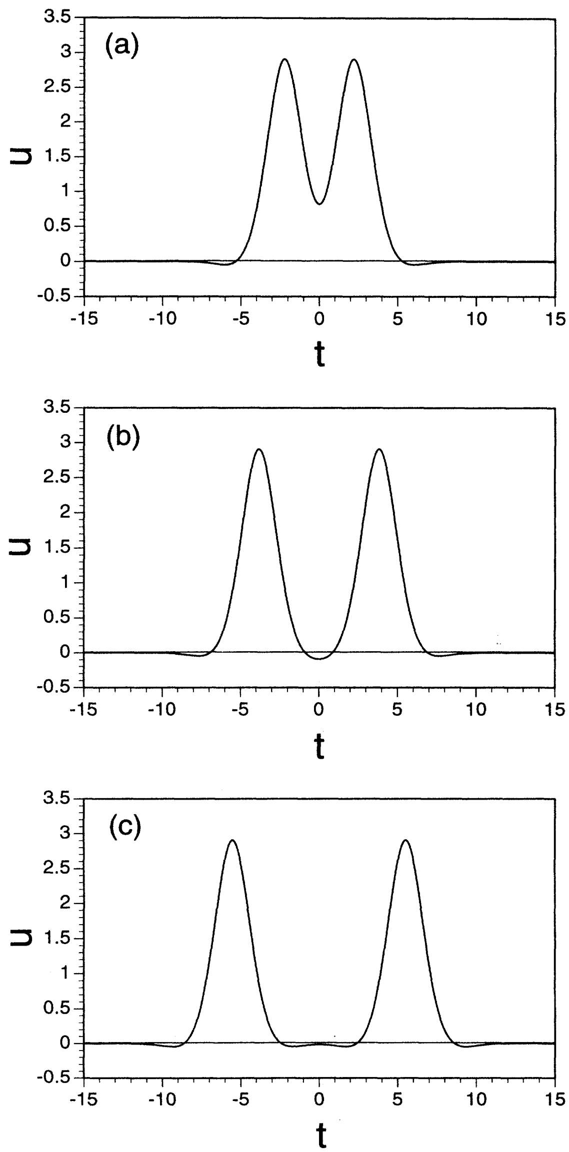

FIG. 4. Examples of symmetric two-soliton BS's: (a) symmetric BS of the first order ($\Delta t = \Delta t_1$, $q = 5.0$, point C in Figs. 2 and 3, $\kappa \approx 0.750$); (b) symmetric BS of the second-order ($\Delta t = \Delta t_2$, $q = 5.0$, point D in Figs. 2, and 3, $\kappa \approx 0.055$); (c) symmetric BS of the third order ($\Delta t = \Delta t_3$, $q = 5.0$, point E in Figs. 2 and 3, $\kappa \approx 0.015$). Points D and E coincide on the scale of Figs. 2 and 3, but they belong to different curves.

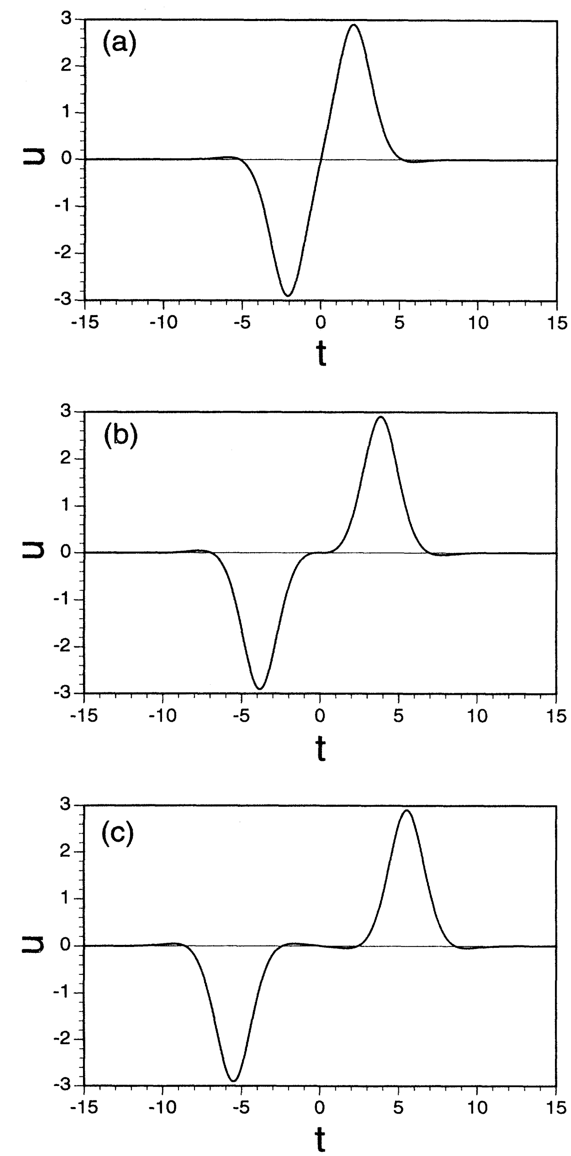

FIG. 5. Examples of antisymmetric two-soliton BS's: (a) antisymmetric BS of the first order ($\Delta t = \Delta t_1$, $q = 5.0$, point F in Figs. 2 and 3, $\kappa \approx 0.603$); (b) antisymmetric BS of the second-order ($\Delta t = \Delta t_2$, $q = 5.0$, point G in Figs. 2 and 3, $\kappa \approx 0.109$); (c) antisymmetric BS of the third order ($\Delta t = \Delta t_3$, $q = 5.0$, point H in Figs. 2 and 3, $\kappa \approx 0.010$). Points G and H coincide in the scale of Figs. 2 and 3, but they belong to different curves.

(3) and (5), respectively.

Each curve in Figs. 2 and 3 corresponds to a family of soliton solutions of Eq. (2). The family of single-soliton solutions exists for every $q > 0$. In the interval $0 < q < 0.25$, one-soliton solutions do not have oscillating tails. On the other hand, for $q > 0.25$, asymptotics of one-soliton solutions are nonmonotonic (oscillating tails). (Two characteristic examples of one-soliton solutions are shown in Fig. 1.) In the range $q > 0.25$ we also have found many families of two-soliton (or multisoliton) BS's. Among two-soliton BS families, there are both symmetric and antisymmetric ones. In Figs. 2 and 3 only curves corresponding to families of two-, three-, and four-soliton BS's are shown, but multisoliton BS's with any number N of partial solitons exist as well. For every number N, there are infinitely many families of BS's, but some of the corresponding curves are located very close to each other and thus cannot be distinguished on the scale of Figs. 2 and 3. Characteristic examples of symmetric two-soliton BS's are shown in Fig. 4 and characteristic examples of antisymmetric two-soliton BS's are shown in Fig. 5.

We have also studied the stability of two-soliton BS's using direct numerical simulations. We used a split-step beam propagation method [22]. In addition to the straightforward propagation of various stationary BS's, we also studied the evolution of perturbation eigenmodes. To do this, we have linearized Eq. (2) around the BS of interest and we have solved the linearized equation to find exponentially growing modes (for details of this technique see, for example, [23]). First of all, we have found that single-soliton solutions are stable for all values of the parameter q ($q > 0$). In other words, all single solitons (with and without oscillating tails) are stable. However, all stationary two-soliton BS's are unstable in accordance with the above analysis. For all two-soliton BS's a mode, growing as $\Delta \sim e^{\kappa x}$ ($\kappa > 0$), always exists. The values of κ for the fastest exponentially growing mode of particular BS are given in the figure captions of Figs. 4 and 5. (All multisoliton BS's are also unstable as a direct consequence of instability of two-soliton BS's.)

BS's that correspond to local maxima of H_{int} are unstable with respect to perturbations that change the relative distances between the centers of the partial solitons.

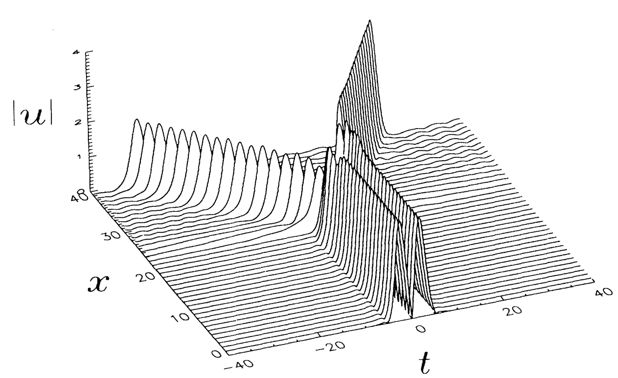

FIG. 6. Evolution of the slightly perturbed two-soliton antisymmetric BS of the first order [$\Delta t = \Delta t_1$, $q = 5.0$, point F in Figs. 2 and 3, the solution in Fig. 5(a)].

BS's that correspond to local minima of H_{int} are also unstable. The exponentially growing perturbation corresponding to this instability keeps the relative distance between the two interacting solitons intact. Instead, it leads to an increase in the amplitude of one soliton and to a decrease in the amplitude of the other one.

A typical example of the latter behavior is shown in Fig. 6. The initial condition for this simulation is chosen in the form of an antisymmetric two-soliton BS corresponding to point F of Fig. 1 (and to a local minimum of H_{int}). First, this BS, which is slightly perturbed by an asymmetric perturbation, propagates some distance without noticeable change of its form. Then a perturbation starts to grow quickly. It does not change the relative distance between the two partial solitons, but it changes the difference between their amplitudes. When this BS evolves far from the initial stationary form, it decays into two single solitons with very different amplitudes which move away from each other. The similar behavior has been observed for the evolution of other slightly perturbed BS's which correspond to local minima of H_{int}. However, we should note that the exponential increment κ of a growing perturbation is very small for BS's of sufficiently high order ($n > 3$). The bound energy of these BS's is also exponentially small ($\sim e^{-\lambda(q)\Delta t_n}$).

VI. CONCLUSION

In conclusion, we have investigated, both analytically and numerically, stationary BS's of solitons in optical fibers with fourth-order dispersion. We have shown that two-soliton (and multisoliton) BS's can exist in the region of parameters where single solitons have oscillating tails. We have also shown that all these two-soliton (and multisoliton) BS's are unstable. A stability criterion for two-soliton BS's, in terms of Hamiltonian versus energy dependence for a family of single soliton solutions, is proposed. This criterion can be used for any dynamical equation which can be described by a conservative Hamiltonian and has a family of one-soliton solutions with oscillating tails.

In spite of the instability of soliton BS's, the idea of using solitons with oscillating tails in telecommunications remains valid since we have shown that oscillatory tails establish a potential barrier in the interaction between two neighboring solitons. This potential barrier exists for equal as well as unequal solitons, provided both solitons have oscillatory tails. Thus solitons cannot approach each other closer than a certain minimal distance and time slots for solitons in communication lines can still be arranged. Investigations of dynamical behavior and interactions of oscillating tail solitons are currently being carried out.

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